Einstein-Podolsky-Rosen-Bohm experiment with massive particles as a test of relativistic center-of-mass position operator

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The nonrelativistic singlet state average $\langle\psi|\vec{a}\cdot\vec{\sigma}\otimes\vec{b}\cdot\vec{\sigma}|\psi\rangle = -\vec{a}\cdot\vec{b}$ can be relativistically generalized if one defines spin *via* the relativistic center-of-mass operator. The relativistic correction is quadratic in v/c and can be measured in Einstein-Podolsky-Rosen-Bohm-type experiments with massive spin-1/2 particles. A deviation from the nonrelativistic formula would indicate that for relativistic nonzero-spin particles centers of mass and charge do not coincide.

When Einstein, Podolsky and Rosen (EPR) formulated in 1935 their famous paradox [1,2], the main problem they addressed was an essentially academic question of completness of the quantum theory. Some three decades later Bell [3] derived an inequality which allowed for relating the EPR *Gedankenexperiment* to a real experimental situation. The photon-pair tests of the Bell inequality performed to date have ruled out a large class of hidden-variable theories. To eliminate some of the remaining possibilities one has to violate the so-called strong Bell inequalities [4,5]. For this reason Fry *et al.* [6] have recently returned to the original idea of Bohm and Bell and propose to test the strong inequalities by using pairs of correlated spin-1/2 *massive* particles. The proposal involves two ^{199}Hg atoms, each with nuclear spin $\frac{1}{2}$, produced in an EPR-Bohm entangled state by dissociation of dimers of the ^{199}Hg$_2$ isotopomer using a spectroscopically selective stimulated Raman process.

In this Letter I want to show that an EPR-Bohm experiment with pairs of *massive* spin-1/2 particles may simultaneously solve another old (in fact, even older than EPR) problem of quantum mechanics.

As is widely known E. Schrödinger in his 1930 paper [7] examined the behavior of the coordinate operator $\boldsymbol{x}$ associated with Dirac's equation and discovered the oscillatory motion he called the *Zitterbewegung*. The *Zitterbewegung* takes place with respect to the *center-of-mass* position operator $\boldsymbol{x}_A$. The operator $\boldsymbol{x}$ is in contemporary literature [8] interpreted as the *center-of-charge* operator, since it is $\boldsymbol{x}$ and not $\boldsymbol{x}_A$ which is used in the minimal electromagnetic coupling. The situation is not typical only of the Dirac equation and is not associated with the presence of negative energy solutions as one is sometimes led to believe. The so-called new Dirac equation generalized by Mukunda *et al.* [9] admits only positive-energy solutions but the *Zitterbewegung* is present and the associated center-of-mass operator is algebraically identical to this implied by Schrödinger's analysis of the Dirac equation (cf. the Barut-Zanghi model of the Dirac electron [10]). The problem is therefore general and is rooted in the structure of the Poincaré group.

In what follows I will use a group representation formulation, elements of which can be found in the 1965 paper by Fleming [11]. The group theoretic approach has the advantage of being applicable to any physical system whose symmetry group is the Poincaré group, or whose symmetry group contains the Poincaré group as a subgroup. The formulation is essentially unrelated to the Dirac equation and can be applied also to hadrons [9].

Let us begin with generators of the unitary, infinite dimensional irreducible representation of the Poincaré group corresponding to a nonzero mass m and spin j. Their standard form is [12]

$$\boldsymbol{J} = \frac{\hbar}{i}\boldsymbol{p}\times\frac{\partial}{\partial\boldsymbol{p}} + \boldsymbol{s}, \tag{1}$$

$$\boldsymbol{K} = \pm\Big(|p_0|\frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{p}} - \frac{\boldsymbol{p}\times\boldsymbol{s}}{mc+|p_0|}\Big), \tag{2}$$

$$\boldsymbol{P} = \boldsymbol{p}, \tag{3}$$

$$P_0 = p_0 = \pm\sqrt{\boldsymbol{p}^2 + m^2c^2}. \tag{4}$$

Here $\boldsymbol{s}$ denotes finite dimensional angular momentum matrices corresponding to the $(2j+1)$-dimensional representation D^j of the rotation group.

The center-of-mass position operator which generalizes to any representation the operator $\boldsymbol{x}_A$ of Schrödinger is

$$\boldsymbol{Q} = -\frac{1}{2}\Big(P_0^{-1}\boldsymbol{K} + \mathbf{K}P_0^{-1}\Big) \tag{5}$$

$$= i\hbar\frac{\partial}{\partial\boldsymbol{p}} - i\hbar\frac{\boldsymbol{p}}{2p_0^2} + \frac{\boldsymbol{p}\times\boldsymbol{s}}{|p_0|(mc+|p_0|)}. \tag{6}$$

This operator extends naturally also to massless fields. Jadczyk and Jancewicz [13] found an interesting argument for its uniqueness in the case of the Maxwell field. Orbital angular momentum and spin corresponding to $\boldsymbol{Q}$ are [14,11]

$$\boldsymbol{L} = \boldsymbol{Q} \times \boldsymbol{P} = \frac{\hbar}{i}\boldsymbol{p} \times \frac{\partial}{\partial \boldsymbol{p}} + \frac{|p_0| - mc}{|p_0|}\Big(\boldsymbol{s} - (\boldsymbol{n} \cdot \boldsymbol{s})\boldsymbol{n}\Big), \tag{7}$$

$$\boldsymbol{S} = \boldsymbol{J} - \boldsymbol{L} = \frac{mc}{|p_0|}\boldsymbol{s} + \Big(1 - \frac{mc}{|p_0|}\Big)(\boldsymbol{n} \cdot \boldsymbol{s})\boldsymbol{n} = \sqrt{1 - \beta^2}\boldsymbol{s}_\perp + (\boldsymbol{n} \cdot \boldsymbol{s})\boldsymbol{n}. \tag{8}$$

$\boldsymbol{s}_\perp$ denotes the projection of $\boldsymbol{s}$ on the plane perpendicular to $\boldsymbol{p}$ and $\beta = |\boldsymbol{v}|/c$, where $\boldsymbol{v} = c\boldsymbol{p}/p_0$ is a velocity of the particle. Projection of spin in a direction given by the unit vector $\boldsymbol{a}$ commutes with the Hamiltonian P_0 and equals

$$\boldsymbol{a} \cdot \boldsymbol{S} = \Big[\frac{mc}{|p_0|}\boldsymbol{a} + \Big(1 - \frac{mc}{|p_0|}\Big)(\boldsymbol{n} \cdot \boldsymbol{a})\boldsymbol{n}\Big] \cdot \boldsymbol{s} = \boldsymbol{\alpha}(\boldsymbol{a}, \boldsymbol{p}) \cdot \boldsymbol{s}. \tag{9}$$

The latter equality defines the vector $\boldsymbol{\alpha}(\boldsymbol{a}, \boldsymbol{p})$ whose length is

$$|\boldsymbol{\alpha}(\boldsymbol{a}, \boldsymbol{p})| = \frac{\sqrt{(\boldsymbol{p} \cdot \boldsymbol{a})^2 + m^2 c^2}}{|p_0|}. \tag{10}$$

The eigenvalues of $\boldsymbol{a} \cdot \boldsymbol{S}$ are therefore

$$\lambda_a = j_3 \hbar |\boldsymbol{\alpha}(\boldsymbol{a}, \boldsymbol{p})| \tag{11}$$

where $j_3 = -j, \ldots, +j$. In the infinite momentum/massless limit the eigenvalues of spin in a direction perpendicular to $\boldsymbol{p}$ vanish, which can be regarded as a consequence of the Lorentz flattenning of the moving particle (in these limits $\boldsymbol{S} = (\boldsymbol{n} \cdot \boldsymbol{s})\boldsymbol{n}$). Projection of spin on the momentum direction is equal to the helicity, i.e. $\boldsymbol{p} \cdot \boldsymbol{S} = \boldsymbol{p} \cdot \boldsymbol{s}$ for any $\boldsymbol{p}$, and $\boldsymbol{S} = \boldsymbol{s}$ in the rest frame ($\boldsymbol{p} = 0$). The definition of spin *via* the relativistic center-of-mass operator can be found already in [7]. Also Mukunda *et al.* [9] noticed that the extended models of hadrons based on the generalized "new" Dirac equation can be correctly interpreted provided one defines spin *via* the relativistic center-of-mass operator (the standard "natural" choice of $\boldsymbol{s}$ leads to physical inconsistencies). Bacry [15] observed that a nonrelativistic limit of $\boldsymbol{x}_A$ leads to a correct form of the spin-orbit interaction in the Pauli equation if one uses potentials $V(\boldsymbol{x}_A)$ instead of $V(\boldsymbol{x})$ [16]; an analogous effect was described in [17] where the internal angular momentum of the *Zitterbewegung* leads to spin with the correct $g = 2$ factor. An algebraic curiosity is the fact that the components of $\boldsymbol{S}$ satisfy an algebra which is $so(3)$ in the rest frame and formally contracts to the Euclidean $e(2)$ in the infinite momentum/massless limit, and thus provides an interesting alternative explanation of the privileged role played by the Euclidean group in the theory of massless fields [18,19].

In spite of all these facts suggestng that both $\boldsymbol{Q}$ and $\boldsymbol{S}$ are natural candidates for physical observables no experimental tests distinguishing them from other definitions of position and spin have been proposed so far.

Consider now two spin-1/2 particles in a singlet state (total helicity equals zero) and propagating in the same direction with identical momenta $\boldsymbol{p}$ (more precisely one should take wave packets in momentum space, but for simplicity assume that they are sufficiently well localized around momenta $\boldsymbol{p}$, so that we can approximate them by plane waves):

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|+1/2, \boldsymbol{p}\rangle|-1/2, \boldsymbol{p}\rangle - |-1/2, \boldsymbol{p}\rangle|+1/2, \boldsymbol{p}\rangle\Big). \tag{12}$$

The kets $|\pm 1/2, \boldsymbol{p}\rangle$ form the *helicity* basis. Consider the binary operators $\hat{\boldsymbol{a}} = \boldsymbol{a} \cdot \boldsymbol{S}/|\lambda_a|$, $\hat{\boldsymbol{b}} = \boldsymbol{b} \cdot \boldsymbol{S}/|\lambda_b|$. The average of the relativistic EPR-Bohm-Bell operator is

$$\langle\psi|\hat{\boldsymbol{a}} \otimes \hat{\boldsymbol{b}}|\psi\rangle = -\frac{\boldsymbol{a} \cdot \boldsymbol{b} - \beta^2 \boldsymbol{a}_\perp \cdot \boldsymbol{b}_\perp}{\sqrt{1 + \beta^2\big[(\boldsymbol{n} \cdot \boldsymbol{a})^2 - 1\big]}\sqrt{1 + \beta^2\big[(\boldsymbol{n} \cdot \boldsymbol{b})^2 - 1\big]}} \tag{13}$$

There are several interesting particular cases of the formula (13). First, if $\boldsymbol{a} = \boldsymbol{a}_\perp$, $\boldsymbol{b} = \boldsymbol{b}_\perp$ then

$$\langle\psi|\hat{\boldsymbol{a}} \otimes \hat{\boldsymbol{b}}|\psi\rangle = -\boldsymbol{a} \cdot \boldsymbol{b} \tag{14}$$

which is the nonrelativistic result. If $\boldsymbol{a} \cdot \boldsymbol{n} \neq 0$, $\boldsymbol{b} \cdot \boldsymbol{n} \neq 0$ then in the ultrarelativistic case $\beta^2 = 1$

$$\langle\psi|\hat{\boldsymbol{a}} \otimes \hat{\boldsymbol{b}}|\psi\rangle = -\frac{(\boldsymbol{a} \cdot \boldsymbol{n})(\boldsymbol{b} \cdot \boldsymbol{n})}{|\boldsymbol{a} \cdot \boldsymbol{n}||\boldsymbol{b} \cdot \boldsymbol{n}|} = \pm 1 \tag{15}$$

independently of the choice of $\boldsymbol{a}$, $\boldsymbol{b}$. It is easy to intuitively understand this result: In the ultrarelativistic limit projections of spin in directions perpendicular to the momentum vanish for both particles and spins are (anti)parallel

to the momentum. The most striking case occurs if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular and the nonrelativistic average is 0. Let $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, $\boldsymbol{a} \cdot \boldsymbol{n} = \boldsymbol{b} \cdot \boldsymbol{n} = 1/\sqrt{2}$. Then

$$\langle \psi | \hat{\boldsymbol{a}} \otimes \hat{\boldsymbol{b}} | \psi \rangle = -\frac{\beta^2}{2 - \beta^2}. \tag{16}$$

This average is 0 in the rest frame ($\beta = 0$) and -1 for $\beta = 1$. Any observable deviation from 0 in an EPR-Bohm type experiment would be an indication that the operators $\boldsymbol{S}$ and $\boldsymbol{Q}$ are physically correct observables and that massive spin-1/2 particles are extended in the sense that centers of mass and charge do not coincide. The components of the center-of-charge operator commute whereas those of $\boldsymbol{Q}$ do not commute for nonzero spins. This means that spinning particles cannot be localized at a point [22]. This interesting property seems unavoidable and can be proved at both quantum and classical levels [9,20]. Its experimental verification could not be without implications for the self-energy and renormalization problems.

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[19] Typically the Euclidean structure of massless fields is interpreted in the language of finite dimensional and nonunitary representations. It can be shown that the $SE(2)$ structure is associated with a cyllindrical geometry: The cyllinder is parallel to momentum and the group contains rotations around and translations along the cyllinder. The translations are gauge transformations of a vector potential. The relativistic spin algebra discussed in this Letter provides an explanation of the contraction in terms of the Lorentz flattenning of an extended particle and the corresponding deformation of the spin algebra. The analogy between the two approaches is however even deeper: The noncommuting position operator allows for localizations of the Maxwell fields on curves [13] and the classical phase space picture derived in [20] leads to world-tubes rather than world-lines. An analogous phenomenon is found in the twistor formulation where instead of world-lines the massless fields are intereperted in terms of the Robinson congruence of twisting null world-lines [21]. The cyllindrical approach is described in Y. S. Kim and M. E. Noz, Phys. Rev. D **15**, 335 (1977); Y. S. Kim, Phys. Rev. Lett. **63**, 348-351 (1989). D. Han, Y. S. Kim, and D. Son, Am. J. Phys. **54**, 818 (1986);D. Han, Y. S. Kim, and D. Son, Phys. Lett. **131B**, 327 (1983); D. Han and Y. S. Kim, Am. J. Phys. **49**, 348 (1981); J. J. van der Bij, H. van Dam, and Y. J. Ng, Physica **116A**, 307 (1982); D. Han, Y. S. Kim, and D. Son, Phys. Rev. D **26**, 3717 (1982); Y. S. Kim and M. E. Noz, Theory and

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